

# CANONICAL FACTORIZATION OF THE QUOTIENT MORPHISM FOR AN AFFINE $\mathbb{G}_a$ -VARIETY

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*To Mikhail Zaidenberg, mentor and friend, on the occasion of his seventieth birthday*

**ABSTRACT.** Working over a ground field of characteristic zero, this paper studies the quotient morphism  $\pi : X \rightarrow Y$  for an affine  $\mathbb{G}_a$ -variety  $X$  with affine quotient  $Y$ . It is shown that the degree modules associated to the  $\mathbb{G}_a$ -action give a uniquely determined sequence of dominant  $\mathbb{G}_a$ -equivariant morphisms,  $X = X_r \rightarrow X_{r-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = Y$ , where  $X_i$  is an affine  $\mathbb{G}_a$ -variety and  $X_{i+1} \rightarrow X_i$  is birational for each  $i \geq 1$ . This is the *canonical factorization* of  $\pi$ . We give an algorithm for finding the degree modules associated to the given  $\mathbb{G}_a$ -action, and this yields the canonical factorization of the quotient morphism. The algorithm is applied to compute the canonical factorization for several examples, including the homogeneous  $(2, 5)$ -action on  $\mathbb{A}^3$ . By a fundamental result of Kaliman and Zaidenberg, any birational morphism of affine varieties is an affine modification, and each mapping in these examples is presented as a  $\mathbb{G}_a$ -equivariant affine modification.

## 1. INTRODUCTION

We assume throughout that  $k$  is a field of characteristic zero and  $\mathbb{G}_a$  is the additive group of  $k$ . A  $k$ -affine  $\mathbb{G}_a$ -variety is an affine  $k$ -variety  $X$  equipped with a regular algebraic action of  $\mathbb{G}_a$ . The  $\mathbb{G}_a$ -actions on  $X$  are in bijective correspondence with the locally nilpotent derivations  $D$  of the coordinate ring  $k[X]$ , and if  $Y$  is the categorical quotient of the given  $\mathbb{G}_a$ -action, then  $k[Y] = \ker D$ , the kernel of the derivation (see [6]). By a fundamental result of Winkelmann [15],  $Y$  is always quasi-affine.

Let  $\pi : X \rightarrow Y$  be the quotient morphism for the given  $\mathbb{G}_a$ -action, and assume that  $Y$  is affine. The main purpose of this paper is to define the canonical factorization of  $\pi$  and to give methods for finding it. The canonical factorization gives critical information about the  $\mathbb{G}_a$ -action. In particular, the degree modules

$$\mathcal{F}_n = \ker D^{n+1}$$

are used to define the canonical factorization, where  $D$  is the locally nilpotent derivation of  $k[X]$  induced by the  $\mathbb{G}_a$ -action. For each  $n \geq 0$ ,  $\mathcal{F}_n$  is a module over  $\mathcal{F}_0 = \ker D$ , and  $k[\mathcal{F}_N] = k[X]$  for some  $N \geq 0$ , since  $k[X]$  is an affine ring. Consider the ascending chain of subalgebras:

$$k[Y] = \mathcal{F}_0 \subset k[\mathcal{F}_1] \subset \cdots \subset k[\mathcal{F}_N] = k[X]$$

From this we obtain a uniquely determined sequence of integers  $n_i$  with  $k[\mathcal{F}_{n_i}] \subsetneq k[\mathcal{F}_{n_{i+1}}]$ . If  $X_i = \text{Spec}(k[\mathcal{F}_{n_i}])$  and  $N = n_r$ , then the corresponding sequence of morphisms

$$X = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \rightarrow \cdots \rightarrow X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = Y$$

is the canonical factorization of  $\pi$ , and  $r$  is the index of the action. *Lemma 4.1* and *Lemma 5.1* show the following.

- (1) For each  $i$ ,  $X_i$  is an affine  $\mathbb{G}_a$ -variety and the morphism  $\pi_i$  is  $\mathbb{G}_a$ -equivariant.
- (2) The morphisms  $\pi_1, \dots, \pi_{r-1}$  are birational.

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This factorization can be described in terms of affine modifications, which were introduced by Zariski as a tool in studying birational correspondences, and further developed by Davis [3, 16]. Geometrically, an affine modification of the affine variety  $X$  is a certain affine open subset  $X' \subset X^*$ , where  $\beta : X^* \rightarrow X$  is a blow-up of  $X$  (see *Sect. 3*). Kaliman and Zaidenberg were the first to systematically apply affine modifications to problems in affine algebraic geometry [8, 11]. They proved that any birational morphism of affine varieties is an affine modification ([11] Thm. 1.1). Moreover, in the case  $X$  is a  $\mathbb{G}_a$ -variety, they give conditions as to when the action lifts to an affine modification  $\beta : X' \rightarrow X$ , i.e.,  $X'$  is a  $\mathbb{G}_a$ -variety and  $\beta$  is  $\mathbb{G}_a$ -equivariant ([11] Cor. 2.3). This is called a  $\mathbb{G}_a$ -equivariant affine modification. Thus, each mapping in the canonical factorization for the quotient map of a  $\mathbb{G}_a$ -action is a  $\mathbb{G}_a$ -equivariant affine modification.

The actions of  $\mathbb{G}_a$  on affine spaces  $X = \mathbb{A}^n$  are of particular interest. The  $\mathbb{G}_a$ -actions on the affine plane  $X = \mathbb{A}^2$  were classified by Rentschler in 1968 [12]. This classification shows that the quotient morphism  $\pi : X \rightarrow Y$  is of the form  $X = Y \times \mathbb{A}^1$ , where  $\pi$  is projection on the first factor. Consequently, the index of any planar  $\mathbb{G}_a$ -action is one, whereas in dimension three, there are  $\mathbb{G}_a$ -actions of index greater than one. Although much is known about  $\mathbb{G}_a$ -actions on  $\mathbb{A}^3$ , their complete classification has not been achieved. It is known that, if  $X \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = Y$  is the canonical factorization for a  $\mathbb{G}_a$ -action on  $X = \mathbb{A}^3$ , then  $Y \cong \mathbb{A}^2$  (due to Miyanishi) and  $X_1 \cong \mathbb{A}^3$  (due to Bonnett and Daigle); see *Sect. 7*. Thus, one is led to study affine threefolds  $X_i$  of *sandwich type*, that is, those admitting birational  $\mathbb{G}_a$ -equivariant morphisms  $\mathbb{A}^3 \rightarrow X_i \rightarrow \mathbb{A}^3$ .

In *Sect. 8*, our methods are applied to compute the canonical factorization for several examples, including actions on  $\mathbb{A}^3$ , where each of the birational maps in the factorization is presented as a  $\mathbb{G}_a$ -equivariant affine modification. It is hoped that canonical factorizations provide a new tool for making progress on the classification of  $\mathbb{G}_a$ -actions on  $\mathbb{A}^3$ . Similarly, a great deal of work has been done on the classification of  $\mathbb{G}_a$ -surfaces (see, for example, [5]), and canonical factorizations should be of interest in this endeavor.

*Thm. 4.4* gives the theoretical basis for an algorithm to calculate the degree modules of a locally nilpotent derivation  $D$  of a commutative  $k$ -domain  $B$  in the case where  $\ker D$  is noetherian. As such, it provides a tool for calculating several related objects, which include the following.

- (1) We obtain a method to find the canonical factorization for the quotient morphism of a  $\mathbb{G}_a$ -action on an affine variety, assuming that the quotient is affine.
- (2) The degree modules  $\mathcal{F}_i$  determine the image ideals  $D^i \mathcal{F}_i$ , so the algorithm gives a way to find generators for these ideals. The plinth ideal  $D\mathcal{F}_1$  is especially important.
- (3) The associated graded ring  $\text{Gr}_D(B)$  induced by the degree function of  $D$  is determined by the degree modules  $\mathcal{F}_n$ , so the algorithm gives a way to find generators for this ring up to degree  $n$  once  $\mathcal{F}_n$  has been calculated.
- (4) A basic problem of locally nilpotent derivations is to find generators for a given kernel  $A \subset B$ . In case  $A$  itself admits a locally nilpotent derivation  $\delta$ , finding the degree modules for  $\delta$  gives a generating set for  $A$ .

**Preliminaries.** We assume throughout that  $k$  is a field of characteristic zero. When working with varieties, we further assume that  $k$  is algebraically closed. If  $B$  is a commutative  $k$ -domain, then  $B^{[n]}$  denotes the polynomial ring in  $n$  variables over  $B$ . Given nonzero  $f \in B$ ,  $B_f$  denotes the localization  $S^{-1}B$  for  $S = \{f^n \mid n \in \mathbb{N}\}$ . If  $X$  is an affine  $k$ -variety, then  $k[X]$  is the coordinate ring of  $X$ . Given an ideal  $I \subset k[X]$ ,  $\mathcal{V}(I) \subset X$  is the zero set of  $I$  in  $X$ . If  $B = k[x_1, \dots, x_n] = k^{[n]}$ , then for each  $i$ ,  $\partial_{x_i}$  denotes the partial derivative  $\partial/\partial x_i$  of  $B$  relative to the given system of coordinates.

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## 2. LOCALLY NILPOTENT DERIVATIONS AND $\mathbb{G}_a$ -ACTIONS

Let  $B$  be a commutative  $k$ -domain.

A **locally nilpotent derivation** of  $B$  is a derivation  $D : B \rightarrow B$  such that, for each  $b \in B$ , there exists  $n \in \mathbb{N}$  (depending on  $b$ ) such that  $D^n b = 0$ . Let  $\ker D$  denote the kernel of  $D$ . The set of locally nilpotent derivations of  $B$  is denoted by  $\text{LND}(B)$ . Note that any  $D \in \text{LND}(B)$  is a  $k$ -derivation.

The study of  $\mathbb{G}_a$ -actions on an irreducible affine  $k$ -variety  $X$  is equivalent to the study of locally nilpotent derivations of the coordinate ring  $k[X]$ . In particular, the action induced by  $D \in \text{LND}(B)$  is given by the exponential map  $\exp(tD)$ ,  $t \in \mathbb{G}_a$ , where the invariant ring  $k[X]^{\mathbb{G}_a}$  equals  $\ker D$ . Conversely, every regular algebraic  $\mathbb{G}_a$ -action on  $X$  is of this form. If  $X$  is a  $\mathbb{G}_a$ -variety, then  $X^{\mathbb{G}_a}$  denotes the set of fixed points of the  $\mathbb{G}_a$ -action on  $X$ , which is defined by the ideal  $(DB)$  generated by the image of  $D$ .

Assume that  $D \in \text{LND}(B)$  and  $A = \ker D$ . An ideal  $I \subset B$  is an **integral ideal** for  $D$  if  $DI \subset I$ . A **slice** for  $D$  is any  $s \in B$  such that  $Ds = 1$ . Note that  $D$  has a slice if and only if the mapping  $D : B \rightarrow B$  is surjective. In this case, the **Dixmier map**  $\pi_s : B \rightarrow A$  given by  $\pi_s(b) = \sum_{i \geq 0} \frac{(-1)^i}{i!} D^i b s^i$  is a surjective homomorphism of  $k$ -algebras.

A **local slice** for  $D$  is any  $r \in B$  such that  $D^2 r = 0$  but  $Dr \neq 0$ . If  $f = Dr$ , then:

$$B_f = A_f[r] = A_f^{[1]}$$

The degree function  $\deg_r$  on  $B_f$  restricts to  $B$ , and this restricted function is denoted by  $\deg_D$ . Any other local slice of  $D$  determines the same degree function, and if  $E \in \text{LND}(B)$  and  $\ker E = A$ , then  $D$  and  $E$  determine the same degree function. So  $\deg_D$  is completely determined by the subring  $A$ .

Since  $A = \{b \in B \mid \deg_D b \leq 0\}$  and  $\deg b \geq 0$  for  $b \neq 0$ ,  $A$  is **factorially closed** in  $B$ , meaning that  $a, b \in A$  whenever  $a, b \in B$  and  $ab \in A \setminus \{0\}$ . It follows that:

$$(1) \quad gB \cap A = gA \quad \text{for all } g \in A$$

**Proposition 2.1. (Generating Principle)** *Let  $r \in B$  be a local slice of  $D$  with  $f = Dr$ . Suppose that  $S \subset B$  is a subalgebra satisfying:*

- (i)  $A[r] \subset S \subset B$
- (ii)  $fB \cap S = fS$

*Then  $S = B$ .*

*Proof.* Let  $b \in B$  be given. Since  $r$  is a local slice, there exists an integer  $n \geq 0$  such that  $f^n b \in A[r]$ . By hypothesis (i),  $f^n b \in S$ . Since  $B$  is a domain, repeated application of hypothesis (ii) shows that  $f^n B \cap S = f^n S$ . Therefore,  $f^n b = f^n s$  for some  $s \in S$ . Since  $B$  is a domain, it follows that  $b = s$ .  $\square$

If  $M \subset B$  is an  $A$ -submodule, then  $M$  is **factorially closed** in  $B$  if  $\alpha, \beta \in M$  whenever  $\alpha, \beta \in B$  and  $\alpha\beta \in M \setminus \{0\}$ . We also need the following.

**Definition 2.2.** *Given a non-empty set  $V \subset B$ ,  $V$  is a  **$D$ -set** if the restriction  $\deg_D : V \rightarrow \mathbb{N} \cup \{-\infty\}$  is injective. Let  $M \subset B$  be a free  $A$ -module. A  **$D$ -basis** of  $M$  is a basis which is a  $D$ -set.*

The following two lemmas are obvious but useful.

**Lemma 2.3.** *Let  $V \subset B$  be a  $D$ -set.*

- (a) *The elements of  $V$  are linearly independent over  $A$ .*
- (b) *If  $b \in B$  and  $\deg_D b > \deg_D v$  for all  $v \in V$ , then  $\cup_{i \geq 0} Vb^i$  is a  $D$ -set.*

**Lemma 2.4.** *Let  $M \subset B$  be a finitely generated  $A$ -module,  $M = \sum_{1 \leq i \leq n} Aw_i$ , and let  $b \in B$  satisfy:*

$$\deg_D b > \max\{\deg_D w_i \mid 1 \leq i \leq n\}$$

*Define the  $A$ -module  $N = \sum_{i \geq 0} Mb^i$ .*

- (a) *If  $M$  is a free  $A$ -module, then  $N$  is a free  $A$ -module.*

(b) If  $M$  admits a  $D$ -basis  $\{v_1, \dots, v_m\}$ , then  $N$  admits a  $D$ -basis given by:

$$\{v_i b^j \mid 1 \leq i \leq m, j \geq 0\}$$

The reader is referred to [6] for further details about locally nilpotent derivations.

### 3. AFFINE MODIFICATIONS

In this section, we follow the notation and terminology of [11]

An **affine triple** over  $k$  is of the form  $(B, I, f)$ , where  $B$  is an affine  $k$ -domain,  $I \subset B$  is a nonzero ideal and  $f \in I$ ,  $f \neq 0$ . For such a triple, define

$$f^{-1}I = \{g \in B_f \mid fg \in I\}$$

where  $B_f$  is the localization of  $B$  at  $f$ .

**Definition 3.1.** Given the affine triple  $(B, I, f)$ , the ring  $B[f^{-1}I]$  is the **affine modification** of  $B$  along  $f$  with **center**  $I$ .

Let  $b_1, \dots, b_s \in B$  be such that  $I = (b_1, \dots, b_s)$ . Then  $B[f^{-1}I] = B[b_1/f, \dots, b_s/f]$ , and  $B[f^{-1}I]$  is an affine domain. Let  $X = \text{Spec}(B)$  and  $X_{(I,f)} = \text{Spec}(B[f^{-1}I])$ .

**Definition 3.2.**  $X_{(I,f)}$  is the **affine modification** of  $X$  along  $f$  with **center**  $I$ . The morphism  $p : X_{(I,f)} \rightarrow X$  induced by the inclusion  $B \subset B[f^{-1}I]$  is the **associated morphism** for the affine modification.

Since  $B_f = B[f^{-1}I]_f$ , we see that the associated morphism  $p : X_{(I,f)} \rightarrow X$  is birational, and that the restriction of  $p$  to the set  $\{f \neq 0\}$  is an isomorphism. The **exceptional divisor**  $E$  of  $X_{(I,f)}$  is defined by the ideal  $IB[f^{-1}I]$ .

Our main interest is in the following fact, due to Kaliman and Zaidenberg.

**Theorem 3.3.** ([11], Thm. 1.1) Any birational morphism of affine varieties is the associated morphism of an affine modification.

#### 3.1. Principal Affine Modifications.

**Definition 3.4.** Let  $(B, I, f)$  be an affine triple, and let  $B' = B[f^{-1}I]$  be the induced affine modification.  $B'$  is a **principal** affine modification of  $B$  if and only if  $B'$  is a principal ring extension of  $B$ , that is,  $B' = B[r]$  for some  $r \in \text{frac}(B)$ . In this case,  $X' = \text{Spec}(B')$  is a **principal** affine modification of  $X = \text{Spec}(B)$ .

**Lemma 3.5.** Let  $(B, I, f)$  be an affine triple, and let  $B' = B[f^{-1}I]$  be the induced affine modification. Then  $B'$  is principal if and only if there exist  $g \in B$  and  $n \geq 0$  such that  $B' = B[(f^n)^{-1}J]$ , where  $J = f^n B + gB$ .

*Proof.* Suppose that  $g \in B$  and  $n \geq 0$  are such that  $B' = B[(f^n)^{-1}J]$  for the ideal  $J = f^n B + gB$ . Then  $B' = B[\frac{g}{f^n}]$ , which is principal.

Conversely, if  $B' = B[r]$ , then, by definition, there exists  $n \geq 0$  such that  $f^n r \in B$ . Set  $g = f^n r$  and  $J = f^n B + gB$ . Then  $B[(f^n)^{-1}J] = B'$ .  $\square$

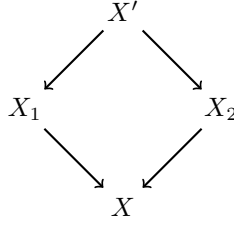
**3.2. Composing Affine Modifications.** Let  $(B, I, f)$  and  $(B, J, g)$  be affine triples. Form the affine modifications:

$$B_1 = B[f^{-1}I], \quad B_2 = B[g^{-1}J], \quad B' = [(fg)^{-1}IJ]$$

Define  $X_1 = \text{Spec}(B_1)$ ,  $X_2 = \text{Spec}(B_2)$  and  $X' = \text{Spec}(B')$ . It is easy to check that:

$$B' = B_1[g^{-1}JB_1] = B_2[f^{-1}IB_2]$$

Therefore, the following diagram of affine modifications commutes.



**3.3. Equivariant Affine Modifications.** Assume that  $X$  is endowed with a  $\mathbb{G}_a$ -action.

**Definition 3.6.** *The affine modification  $X_{(I,f)}$  of  $X$  is  $\mathbb{G}_a$ -equivariant if the  $\mathbb{G}_a$ -action on  $X$  lifts to  $X_{(I,f)}$ , i.e.,  $X_{(I,f)}$  admits a  $\mathbb{G}_a$ -action for which the associated morphism is  $\mathbb{G}_a$ -equivariant.*

Note that, if  $X_{(I,f)}$  admits a  $\mathbb{G}_a$ -action for which the associated morphism is  $\mathbb{G}_a$ -equivariant, then the action is uniquely determined.

Suppose that  $D \in \text{LND}(B)$ ,  $f \in \ker D$  is nonzero, and  $I \subset B$  is a nonzero integral ideal for  $D$  (i.e.,  $DI \subset I$ ). Let  $D'$  be the extension of  $D$  to  $B_f$ . Then  $D'$  is locally nilpotent and:

$$D'(f^{-1}I) = f^{-1}D'I \subset f^{-1}I$$

It follows that  $D'$  restricts (and  $D$  extends) to  $B[f^{-1}I]$ . We have thus shown:

**Theorem 3.7.** ([11], Cor.2.3) *Let  $\rho : \mathbb{G}_a \times X \rightarrow X$  be a  $\mathbb{G}_a$ -action and let  $X_{(I,f)}$  be an affine modification of  $X$  along  $f$  with center  $I$ . If  $f \in k[X]^{\mathbb{G}_a}$  and  $\rho$  restricts to an action on  $\mathcal{V}(I)$ , then  $X_{(I,f)}$  is a  $\mathbb{G}_a$ -equivariant affine modification.*

**Example 3.8.** Consider the affine plane  $X = \mathbb{A}^2$  and its coordinate ring  $B = k[x, y] = k^{[2]}$ . Let  $L, M \subset X$  be the lines defined by  $x = 0$  and  $y = 0$ , respectively, and let  $P$  be their intersection. The locally nilpotent derivation  $D = x \frac{\partial}{\partial y}$  of  $B$  induces a  $\mathbb{G}_a$ -action on  $X$ , and  $L = X^{\mathbb{G}_a}$ . The one-dimensional orbits are lines  $x = x_0$  for  $x_0 \neq 0$ .

Let  $\beta : X^* \rightarrow X$  be the blow-up of  $X$  at  $P$ , let  $E \subset X^*$  be the exceptional divisor over  $P$ , and let  $L^*, M^* \subset X^*$  be the strict transforms of  $L$  and  $M$ , respectively. Let  $U \subset X^*$  be the open set  $U = X^* \setminus L^*$  and let  $E_0 = E \cap U$ . Then  $U \cong \mathbb{A}^2$ , and the composition  $U \hookrightarrow X^* \xrightarrow{\beta} X$  is a birational endomorphism of  $\mathbb{A}^2$ .

As an affine modification, the coordinate ring  $k[U]$  is  $B[x^{-1}I] = k[x, \frac{y}{x}]$ , where  $I \subset B$  is the ideal  $I = xB + yB$  defining  $P$ . Note that  $DI \subset I$ . The morphism  $U \rightarrow X$  is defined by the inclusion  $k[x, y] \hookrightarrow k[x, \frac{y}{x}]$ , with exceptional divisor  $E_0$ .  $D$  extends to the derivation  $D'$  on  $k[x, \frac{y}{x}]$  defined by  $D'x = 0$ ,  $D'(\frac{y}{x}) = 1$ . Thus, the  $\mathbb{G}_a$ -action on  $X$  lifts to a free  $\mathbb{G}_a$ -action on  $U$ , and the associated morphism  $U \rightarrow X$  is  $\mathbb{G}_a$ -equivariant.

The situation is depicted in Fig. 1.

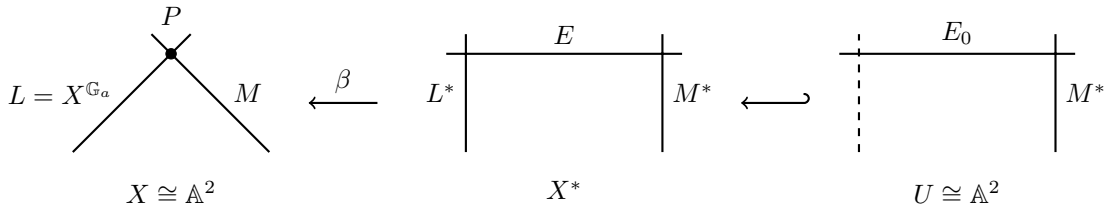


FIGURE 1. A birational  $\mathbb{G}_a$ -endomorphism of  $\mathbb{A}^2$

## 4. DEGREE MODULES

Assume that  $B$  is a commutative  $k$ -domain and that  $A \subset B$  is a subalgebra such that  $A = \ker D$  for some nonzero  $D \in \text{LND}(B)$ . Our primary interest is in the following three related objects.

- (1) The ascending  $\mathbb{N}$ -filtration of  $B$  by  $A$ -modules given by:

$$B = \bigcup_{n \geq 0} \mathcal{F}_n \quad \text{where} \quad \mathcal{F}_n = \ker D^{n+1} = \{f \in B \mid \deg_D f \leq n\}$$

The modules  $\mathcal{F}_n$  are the **degree modules** associated to  $D$ .

- (2) The descending  $\mathbb{N}$ -filtration of  $A$  by ideals given by:

$$A = I_0 \supset I_1 \supset I_2 \supset \cdots \quad \text{where} \quad I_n = A \cap D^n B = D^n \mathcal{F}_n$$

The ideals  $I_n$  are the **image ideals** associated to  $D$ . The **plinth ideal** for  $D$  is  $\text{pl}(D) = I_1$ .

- (3) The ring

$$\text{Gr}_D(B) = \bigoplus_{n \geq 0} I_n \cdot t^n \subset A[t] \cong A^{[1]}$$

is the **associated graded ring** defined by  $D$ .

Observe the following.

- (a) Each  $\mathcal{F}_n$  is a factorially closed  $A$ -submodule of  $B$ .
- (b) The definition of  $\mathcal{F}_n$  depends only on  $A$ , not on the particular derivation  $D$ .
- (c) Given integers  $n, i$  with  $1 \leq i \leq n$ , the following sequence of  $A$ -modules is exact.

$$0 \rightarrow \mathcal{F}_{i-1} \hookrightarrow \mathcal{F}_n \xrightarrow{D^i} D^i \mathcal{F}_n \rightarrow 0$$

In particular,  $I_n \cong \mathcal{F}_n / \mathcal{F}_{n-1}$ .

- (d) Let  $r \in B$  be a local slice for  $D$ . For each  $n \geq 0$ , define the submodule  $\mathcal{G}_n(r) \subset \mathcal{F}_n$  by:

$$\mathcal{G}_n(r) = A[r] \cap \mathcal{F}_n = A \oplus Ar \oplus \cdots \oplus Ar^n$$

If  $r$  is a slice for  $D$ , then  $\mathcal{G}_n(r) = \mathcal{F}_n$  for each  $n \geq 0$ .

**Lemma 4.1.** *If  $A$  is a noetherian ring, then  $\mathcal{F}_n$  is a noetherian  $A$ -module for each  $n \geq 0$ .*

*Proof.*  $\mathcal{F}_0 = A$  is noetherian by hypothesis. Given  $n \geq 1$ , assume by induction that  $\mathcal{F}_m$  is noetherian for  $0 \leq m \leq n-1$ . By the inductive hypothesis,  $\mathcal{F}_{i-1}$  and  $\mathcal{F}_{n-i}$  are noetherian for  $1 \leq i \leq n$ . Therefore, the submodule  $D^i \mathcal{F}_n$  of  $\mathcal{F}_{n-i}$  is also noetherian. Since the sequence (3) above is exact, and since  $\mathcal{F}_{i-1}$  and  $D^i \mathcal{F}_n$  are noetherian, it follows that  $\mathcal{F}_n$  is noetherian.  $\square$

**Example 4.2.** Assume that  $A$  is a noetherian ring. Then there exist an integer  $m \geq 1$  and  $a_1, \dots, a_m \in A$  such that  $\text{pl}(D) = a_1 A + \cdots + a_m A$ . Let  $r_1, \dots, r_m \in \mathcal{F}_1$  be such that  $Dr_i = a_i$  for  $1 \leq i \leq m$ . Given  $s \in \mathcal{F}_1$ , write  $Ds = c_1 a_1 + \cdots + c_m a_m$  for  $c_i \in A$ . Then  $s - (c_1 r_1 + \cdots + c_m r_m) \in A$ . It follows that:

$$\mathcal{F}_1 = A + Ar_1 + \cdots + Ar_m$$

See [1], Lemma 2.2.

Fix a local slice  $r \in B$  and integer  $n \geq 1$ , and set  $f = Dr \in A$ . Suppose that  $M_0$  is an  $A$ -submodule of  $B$  such that  $\mathcal{G}_n(r) \subset M_0 \subset \mathcal{F}_n$ . Inductively, define the ascending chain  $M_0 \subset M_1 \subset M_2 \subset \cdots$  of  $A$ -submodules of  $B$  by:

$$M_i = \{h \in B \mid fh \in M_{i-1}\} = \{h \in B \mid f^i h \in M_0\} \quad (i \geq 1)$$

Then  $fM_{i+1} \subset M_i \subset M_{i+1} \subset \mathcal{F}_n$  for each  $i \geq 0$ , since  $\mathcal{F}_n$  is factorially closed.

**Lemma 4.3.**  $\mathcal{F}_n = \bigcup_{i \geq 0} M_i$

*Proof.* It must be shown that, to each  $h \in \mathcal{F}_n$ , there exists  $s \geq 0$  such that  $h \in M_s$ . Let  $s \geq 0$  be such that  $f^s h \in A[r]$ . Then  $f^s h \in A[r] \cap \mathcal{F}_n = \mathcal{G}_n(r) \subset M_0$ . By definition of the modules  $M_i$ , we see that  $h \in M_s$ , and the lemma is proved.  $\square$

**Theorem 4.4.** *The following conditions are equivalent.*

- (1)  $fB \cap M_s = fM_s$  for some  $s \geq 0$ .
- (2)  $M_s = M_{s+1}$  for some  $s \geq 0$ .
- (3) The ascending chain  $M_0 \subset M_1 \subset M_2 \subset \dots$  stabilizes.
- (4)  $\mathcal{F}_n = M_s$  for some  $s \geq 0$ .

If  $A$  is a noetherian ring, then these conditions are valid.

*Proof.* (1)  $\Leftrightarrow$  (2): This follows by definition of the modules  $M_i$ .

(2)  $\Rightarrow$  (3): Assume that  $M_s = M_{s+1}$  for some  $s \geq 0$ . If  $h \in M_{s+2}$ , then:

$$fh \in M_{s+1} = M_s \Rightarrow h \in M_{s+1} = M_s$$

Therefore,  $M_s = M_{s+2}$ . By induction, we obtain that  $M_s = M_S$  for all  $S \geq s$ .

(3)  $\Rightarrow$  (4): Assume that, for some  $s \geq 0$ ,  $M_s = M_S$  for all  $S \geq s$ . By Lemma 4.3, it follows that  $\mathcal{F}_n = \cup_{i \geq 0} M_i = M_s$ .

(4)  $\Rightarrow$  (2): Assume that  $\mathcal{F}_n = M_s$  for some  $s \geq 0$ . Then  $M_{s+1} \subset \mathcal{F}_n = M_s \subset M_{s+1}$  implies that  $M_s = M_{s+1}$ .

We have thus shown that conditions (1)-(4) are equivalent. Assume that  $A$  is a noetherian ring. By Lemma 4.1, there exists a finite module basis  $\{z_1, \dots, z_t\}$  for  $\mathcal{F}_n$ , where  $t \geq 1$ . By Lemma 4.3, there exists  $s \geq 0$  such that  $\{z_1, \dots, z_t\} \subset M_s$ . Therefore,  $\mathcal{F}_n = M_s$ , and condition (4) is validated.  $\square$

Theorem 4.4 gives the theoretical basis for an algorithm to calculate the degree modules  $\mathcal{F}_n$  in the case where  $A$  is noetherian. Suppose that  $\{x_1, \dots, x_m\}$  is a set of module generators for  $M_i$  and let  $\{X_1, \dots, X_m\}$  be a basis for the free  $A$ -module of rank  $m$ . Define  $\rho : A^m \rightarrow M_i$  by  $\rho(X_j) = x_j$ . Then  $K := \rho^{-1}(fB \cap M_i)$  is a submodule of  $A^m$ . Since  $A$  is noetherian,  $K$  is finitely generated. Generators for  $K$  can be calculated by standard methods. Suppose that  $\{Y_1, \dots, Y_l\}$  is a set of generators for  $K$ , and let  $s_1, \dots, s_l \in B$  be such that  $\rho(Y_j) = fs_j$ . Then  $M_{i+1} = M_i + As_1 + \dots + As_l$ .

## 5. DEGREE RESOLUTIONS

We continue the notation and assumptions of the preceding section.

**5.1. The Subrings  $k[\mathcal{F}_n]$ .** Define subrings  $B_i = k[\mathcal{F}_i] \subset B$ ,  $i \geq 0$ . Then  $B_0 = A$  and  $B_i \subset B_{i+1}$  for  $i \geq 0$ . If  $B$  is  $G$ -graded by an abelian group  $G$  and  $A$  is a  $G$ -graded subalgebra, then each  $\mathcal{F}_i$  is a  $G$ -graded submodule and each  $B_i$  is a  $G$ -graded subalgebra.

**Lemma 5.1.** *Let  $i$  be an integer,  $i \geq 0$ .*

- (a)  $D$  restricts to  $D_i : B_i \rightarrow B_i$ , where  $A = \ker D_i$
- (b) If  $i \geq 1$ , then  $\text{frac}(B_i) = \text{frac}(B)$

*Proof.* Given  $i \geq 1$ , the definition of  $\mathcal{F}_i$  implies that  $D(\mathcal{F}_i) \subset \mathcal{F}_{i-1} \subset \mathcal{F}_i$ . Since  $D$  restricts to a generating set for  $B_i$ , it follows that  $D$  restricts to  $B_i$ , and part (a) is confirmed.

Part (b) follows from the observation that  $S^{-1}B_i = S^{-1}B = S^{-1}A[r]$  for  $S = A \setminus \{0\}$  and some  $r \in \mathcal{F}_1$ .  $\square$

**5.2. Degree Resolutions for Affine Rings.** Assume that  $B$  is an affine  $k$ -domain. In this case,  $B_N = B$  for some  $N \geq 0$ . It is possible that  $B_i = B_{i+1}$  for some  $i$ . Let  $n_i$ ,  $0 \leq i \leq r$ , be the unique subsequence of  $0, 1, \dots, N$  such that:

$$\{B_0, \dots, B_N\} = \{B_{n_0}, \dots, B_{n_r}\} \quad \text{and} \quad B_{n_{i-1}} \subsetneq B_{n_i} \subsetneq B_{n_{i+1}} \quad \text{for } 1 \leq i < r$$

Note that, when  $D \neq 0$ ,  $n_0 = 0$ ,  $n_1 = 1$  and  $B_{n_r} = B$ . Let  $\mathcal{N}_B(A) = \{0, 1, n_2, \dots, n_r\}$ . Both the integer  $r$  and the sequence of subrings

$$(2) \quad A = B_0 \subset B_1 \subset B_{n_2} \subset \dots \subset B_{n_r} = B$$

are uniquely determined by  $A$ .

**Definition 5.2.** *The sequence of inclusions (2) is the **degree resolution** of  $B$  over  $A$ . The integer  $r$  is the **index** of  $A$  in  $B$ , denoted  $\text{index}_B(A)$ ; we also say that  $r$  is the **index** of  $D$ .*

We make the following observations.

- (a)  $\text{index}_B(A) + 1 = |\mathcal{N}_B(A)|$
- (b)  $\text{index}_{B_{n_i}}(A) = i + 1$
- (c)  $\text{index}_B(A) = 0$  if and only if  $A = B$  if and only if  $D = 0$
- (d) If  $D$  has a slice, then  $\text{index}_B(A) = 1$

Let  $R \subset B$  be an affine subring such that  $D$  restricts to  $R$ . The induced filtration of  $R$  is  $R = \bigcup_{i \geq 0} R \cap \mathcal{F}_i$  and if  $R_{n_i} = R \cap B_{n_i}$  for  $n_i \in \mathcal{N}_B(A)$ , then the degree resolution of  $R$  over  $R \cap A$  is a refinement of the sequence:

$$R \cap A = R_0 \subset R_1 \subset R_{n_2} \subset \dots \subset R_{n_r} = R$$

Therefore,  $\mathcal{N}_R(R \cap A) \subset \mathcal{N}_B(A)$  and  $\text{index}_R(R \cap A) \leq \text{index}_B(A)$ .

**Example 5.3.** Let  $B = k^{[2]}$  and let  $D \in \text{LND}(B)$  be nonzero. By Rentschler's Theorem, there exist  $x, y \in B$  such that  $A = \ker D = k[x]$ ,  $Dy \in k[x]$  and  $B = k[x, y]$ . See [6], Thm. 4.1. Therefore,  $\mathcal{F}_n = \mathcal{G}_n(y) = A \oplus Ay \oplus \dots \oplus Ay^n$  for each  $n \geq 0$ . In particular, every nonzero element of  $\text{LND}(B)$  has index one.

**Example 5.4.** Let  $B = k[x, y, z] = k^{[3]}$ . Define  $P, Q \in B$  by  $Q = xz + y^2$  and  $P = y + Q^2$ , and define  $D \in \text{LND}(B)$  by:

$$D = P_z \partial_y - P_y \partial_z = 2xQ \partial_y - (1 + 4yQ) \partial_z$$

Then  $A = \ker D = k[x, P]$  and  $DQ = x$ . Define  $J \subset \mathbb{Z}^2$  by:

$$J = \{(i, j) \mid 0 \leq i \leq 3, j \geq 0\}$$

Given  $n \geq 0$ , define  $J_n \subset J$  by:

$$J_n = \{(i, j) \in J \mid i + 4j \leq n\}$$

We will show:

$$(3) \quad B = \bigoplus_{(i,j) \in J} A Q^i z^j \quad \text{and} \quad \mathcal{F}_n = \bigoplus_{(i,j) \in J_n} A Q^i z^j$$

Since  $y \in k[P, Q]$  we have  $B = k[x, P, Q, z] = A[z, Q]$ . In addition, the equality

$$xz = Q - (P - Q^2)^2$$

shows that  $Q$  is integral of degree 4 over  $A[z] \cong_A A^{[1]}$ . Therefore:

$$B = A[z, Q] = A[z] \oplus A[z]Q \oplus A[z]Q^2 \oplus A[z]Q^3 = \bigoplus_{(i,j) \in J} A Q^i z^j$$

This shows the first equality of (3). Since  $\deg_D Q = 1$  and  $\deg_D z = 4$ , the degrees  $\deg_D(Q^i z^j)$  for  $(i, j) \in J$  are distinct and  $\{Q^i z^j \mid (i, j) \in J\}$  is a  $D$ -basis for  $B$ , which implies the second equality of (3). Therefore,  $\mathcal{N}_B(A) = \{0, 1, 4\}$  and  $\text{index}_B(A) = 2$ .



**Example 5.5.** Let  $B = k[x_1, x_2, y_1, y_2] = k^{[4]}$  and define  $T \in \text{LND}(B)$  by:

$$T = x_1 \partial_{y_1} + x_2 \partial_{y_2}$$

Then  $A = \ker T = k[x_1, x_2, g]$  where  $g = x_1 y_2 - x_2 y_1$ . Since  $y_1$  and  $y_2$  are local slices,  $B = B_1 = k[\mathcal{F}_1]$ , and  $\text{index}_B(A) = 1$ . We calculate  $\mathcal{F}_1$ .

Define  $M \subset \mathcal{F}_1$  by  $M = A + Ay_1 + Ay_2$ . Suppose that  $x_1 w \in M$  for  $w \in \mathcal{F}_1$ , and write  $x_1 w = a_0 + a_1 y_1 + a_2 y_2$  for  $a_i \in A$ . Then  $x_1 Dw = a_1 x_1 + a_2 x_2$ , which implies that  $a_2 \in x_1 B \cap A = x_1 A$  and  $a_0 + a_1 y_1 \in x_1 B$ .

Let  $p : B \rightarrow B/x_1 B$  be the canonical surjection, let  $\bar{b} = p(b)$  for  $b \in B$ , and let  $\bar{A} = p(A) = k[\bar{x}_2, \bar{g}]$ , noting that  $\bar{g} + \bar{x}_2 \bar{y}_1 = 0$ . If  $\bar{A}X \oplus \bar{A}Y$  is the free  $\bar{A}$ -module of rank 2, then:

$$\bar{A} + \bar{A}\bar{y}_1 = \bar{A}X \oplus \bar{A}Y / \bar{A}(\bar{g}X + \bar{x}_2 Y)$$

Therefore,  $\bar{a}_0 + \bar{a}_1 \bar{y}_1 = 0$  implies  $a_0 + a_1 y_1 = \alpha(g + x_2 y_1) = \alpha x_1 y_2$  for some  $\alpha \in A$ . So  $w \in M$  and  $M = \mathcal{F}_1$ . In addition, the plinth ideal  $D\mathcal{F}_1$  equals  $x_1 A + x_2 A$ . Note that, unlike in the preceding examples,  $\mathcal{F}_1$  is not a free  $A$ -module.

## 6. CANONICAL FACTORIZATIONS

We continue the notation and assumptions of the preceding section, with the added assumptions that  $k$  is algebraically closed, and that both  $A$  and  $B$  are  $k$ -affine. In this case, the geometric content of *Lemma 5.1* is as follows.

Let  $X = \text{Spec}(B)$  and  $Y = \text{Spec}(A)$ , and let  $\pi : X \rightarrow Y$  be the quotient map for the  $\mathbb{G}_a$ -action on  $X$  determined by  $D$ . By *Lemma 4.1*,  $B_{n_i}$  is affine for each  $n_i \in \mathcal{N}_B(A)$ . Define  $X_i = \text{Spec}(B_{n_i})$ ,  $0 \leq i \leq r$ .

For  $0 \leq i \leq r-1$ , the inclusion  $B_{n_i} \rightarrow B_{n_{i+1}}$  induces a dominant  $\mathbb{G}_a$ -equivariant morphism  $\pi_i : X_{i+1} \rightarrow X_i$  which is birational if  $i \neq 0$ . Therefore,  $\pi$  factors into the uniquely determined sequence of dominant  $\mathbb{G}_a$ -equivariant morphisms

$$(4) \quad X = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \rightarrow \cdots \rightarrow X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = Y$$

where each morphism  $\pi_{r-1}, \dots, \pi_1$  is birational.

**Definition 6.1.** *The sequence of mappings (4) is the **canonical factorization** of the quotient morphism  $\pi$  for the  $\mathbb{G}_a$ -action determined by  $D$ . The integer  $r$  is the **index** of the  $\mathbb{G}_a$ -action.*

From *Thm. 3.3* and *Thm. 3.7*, we conclude that the maps  $\pi_1, \dots, \pi_{r-1}$  in the canonical factorization (4) form a sequence of  $\mathbb{G}_a$ -equivariant affine modifications. Regarding fixed points, note that  $\pi_{i-1}(X_i^{\mathbb{G}_a}) \subset X_{i-1}^{\mathbb{G}_a}$  for each  $i$  with  $1 \leq i \leq r$ . Note also that, for  $1 \leq i \leq r$ , the  $\mathbb{G}_a$ -action on  $X_i$  has index  $i$ , and its canonical factorization is given by  $\pi_0 \pi_1 \cdots \pi_{i-1}$ .

## 7. $\mathbb{G}_a$ -ACTIONS ON $\mathbb{A}^3$

Suppose that  $\rho : \mathbb{G}_a \times \mathbb{A}^3 \rightarrow \mathbb{A}^3$  is a  $\mathbb{G}_a$ -action defined by the locally nilpotent derivation  $D$  of  $k^{[3]}$ . Let  $A = \ker D$  and  $Y = \text{Spec}(A)$ , and let  $\pi : X \rightarrow Y$  be the quotient morphism for  $\rho$ . The following properties are known.

- (a)  $A \cong k^{[2]}$ , or equivalently,  $Y \cong \mathbb{A}^2$  (due to Miyanishi).
- (b)  $\pi$  is surjective (due to Bonnett).
- (c) The plinth ideal  $I_1 = A \cap DB$  is a principal ideal of  $A$  (due to Daigle). Equivalently, there exists a local slice  $r$  of  $D$  for which  $\mathcal{F}_1 = A \oplus Ar$ .
- (d) If  $\rho$  is fixed-point free, then  $\rho$  is a translation, i.e., given by  $\rho(t, (x, y, z)) = (x, y, z + t)$  for some coordinates  $(x, y, z)$  on  $\mathbb{A}^3$  (due to Kaliman).

None of these properties generalizes to higher dimensional affine spaces. See [6], Chap. 5 for details about these results.

Suppose that the canonical factorization of  $\pi$  is given as in line (4) above. Let  $C \subset Y$  be the curve defined by  $I_1$ , which is, in general, reducible. We have:

- (e) Each irreducible component of  $C$  is a polynomial curve ([9], Thm. 5.2).
- (f)  $X_1 \cong Y \times \mathbb{A}^1 \cong \mathbb{A}^3$  and  $\pi_0 : X_1 \rightarrow Y$  is projection on the first factor.
- (g) Given  $p \in Y \setminus C$ , the fiber  $\pi_0^{-1}(p) \cong \mathbb{A}^1$  is a single orbit.
- (h)  $X_1^{\mathbb{G}_a} \subset \pi_0^{-1}(C)$
- (i) The mapping  $\pi_1 \cdots \pi_{r-1}$  is a  $\mathbb{G}_a$ -equivariant birational endomorphism of  $\mathbb{A}^3$

In particular, part (f) classifies the  $\mathbb{G}_a$ -actions on  $\mathbb{A}^3$  of index one.

Some of the examples presented in the next section use the following fact. Let  $L \subset H \subset X = \mathbb{A}^3$ , where  $H = \mathbb{A}^2$  is a coordinate plane and  $L = \mathbb{A}^1$  is a coordinate line. Let  $\beta : X^* \rightarrow X$  be the blow-up of  $X$  along  $L$ , and let  $H^* \subset X^*$  be the proper transform of  $H$ . If  $U \subset X^*$  is the open subset  $U = X^* \setminus H^*$ , then  $U \cong \mathbb{A}^3$  and  $U \hookrightarrow X^* \xrightarrow{\beta} X$  is a birational endomorphism of  $\mathbb{A}^3$ .

## 8. EXAMPLES

**8.1. The (1, 2) Action on  $\mathbb{A}^3$ .** Let  $B_1 = k[x, y, z] = k^{[3]}$  and define the derivation  $D_1$  of  $B$  by  $D_1 = x \frac{\partial}{\partial y}$ . Let  $X_1 = \mathbb{A}^3$  and let  $H \subset X_1$  be the plane defined by  $x = 0$ . Then  $X_1^{\mathbb{G}_a} = H$  for the  $\mathbb{G}_a$ -action on  $X_1$  defined by  $D_1$ . The kernel of  $D_1$  is  $A = k[x, z]$ , and if  $Y = \text{Spec}(A)$ , then the quotient map  $\pi_0 : X_1 \rightarrow Y$  is a standard projection of  $\mathbb{A}^3$  onto  $\mathbb{A}^2$ .

Let  $C \subset H$  be the curve (a coordinate line) defined by  $x = z + y^2 = 0$ . Let  $\beta : X_1^* \rightarrow X_1$  be the blow-up of  $X_1$  along  $C$ , let  $E \subset X_1^*$  be the exceptional divisor lying over  $C$ , and let  $H^* \subset X_1^*$  be the strict transform of  $H$ . If  $X_2 \subset X_1^*$  is the open subset  $X_2 = X_1^* \setminus H^*$ , then (as observed above)  $X_2 \cong \mathbb{A}^3$  and the mapping

$$\pi_1 : X_2 \hookrightarrow X_1^* \xrightarrow{\beta} X_1$$

is a birational endomorphism of  $\mathbb{A}^3$ . Since  $C \subset X_1^{\mathbb{G}_a}$ , the  $\mathbb{G}_a$ -action on  $X_1$  lifts to  $X_2$  and  $\pi_1$  is equivariant.

The curve  $C$  is defined by the ideal  $I = xB_1 + (z + y^2)B_1$ , where  $D_1 I \subset I$ . If  $B_2 = k[X_2]$ , then  $B_2 = B_1[x^{-1}I] = k[x, y, u]$ , where  $u = \frac{z+y^2}{x}$ . If  $D_2$  is the extension of  $D_1$  to  $B_2$ , then:

$$D_2 x = 0, \quad D_2 y = x, \quad D_2 u = 2y$$

Moreover,  $A = \ker D_2 = k[x, z] = k[x, xu - y^2]$ .  $D_2$  is the homogeneous (1, 2) derivation of  $k^{[3]}$ , which is of index 2; see [6], 5.1.5.

The canonical factorization of the quotient morphism  $\pi : X_2 \rightarrow Y$  is given by:

$$X = X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = Y$$

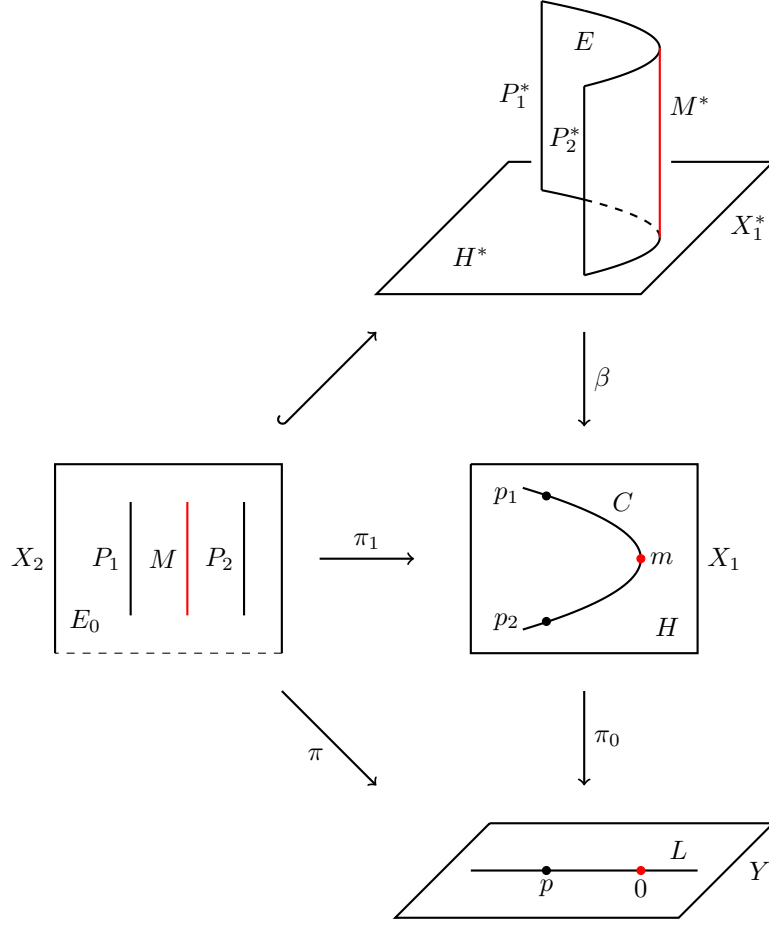
Let  $E_0 \subset X_2$  be the plane defined by  $x = 0$ , noting that  $M = X_2^{\mathbb{G}_a} \subset E_0$  is the line defined by  $x = y = 0$ . Let  $L \subset Y$  be the line defined by  $x = 0$ . Since  $(B_2)_x = (B_1)_x = A_x[y]$ , it follows that  $\pi_1 : X_2 \setminus E_0 \rightarrow X_1 \setminus H$  is an isomorphism, and  $\pi_0^{-1}(q)$  is a single orbit (isomorphic to  $\mathbb{A}^1$ ) for each  $q \in Y \setminus L$ .

Consider the restriction:

$$\pi : E_0 \cong \mathbb{A}^2 \xrightarrow{\pi_1} H \cong \mathbb{A}^2 \xrightarrow{\pi_0} L \cong \mathbb{A}^1$$

Given  $p \in L$ ,  $\pi^{-1}(p)$  is a union  $P_1 \cup P_2$  of two lines in  $E_0$  which are orbits in  $X_2$  if  $p \neq 0$ , and  $\pi^{-1}(0) = M$ . The situation is depicted in *Fig. 2*.

**8.2. The (2, 5) Action on  $\mathbb{A}^3$ .** This example is considerably more complicated than the preceding example. It is of rank three, meaning that the ring of invariants for the  $\mathbb{G}_a$ -action on  $\mathbb{A}^3$  does not contain a variable.

FIGURE 2. Canonical Factorization for the  $(1, 2)$   $\mathbb{G}_a$ -Action on  $\mathbb{A}^3$ 

8.2.1. *The  $(2, 5)$  Derivation.* The standard  $\mathbb{Z}$ -grading of the polynomial ring  $B = k[x, y, z] = k^{[3]}$  is defined by letting  $x, y, z$  be homogeneous of degree one. Define homogeneous elements  $F, G, R, S \in B$  by:

$$F = xz - y^2, \quad G = zF^2 + 2x^2yF + x^5, \quad R = x^3 + yF, \quad S = x^2y + zF$$

Observe the following relations.

$$F^3 + R^2 = xG, \quad x^2R + FS = G, \quad xS - yR = F^2$$

The homogeneous  $k$ -derivation  $D$  of  $B$  defined by the jacobian determinant

$$Dh = \frac{\partial(F, G, h)}{\partial(x, y, z)} \quad (h \in B)$$

is locally nilpotent, and if  $A = \ker D$ , then  $A = k[F, G]$ ; see [6], 5.4.  $D$  is the homogeneous  $(2, 5)$  derivation of  $B$  and the corresponding  $\mathbb{G}_a$ -action is the  $(2, 5)$   $\mathbb{G}_a$ -action on  $\mathbb{A}^3$ . Note the following images.

$$DR = -FG, \quad Dx = -2FR, \quad DS = x(5xG - 4F^3), \quad Dy = 6x^2R - G, \quad Dz = 2x(5yR + F^2)$$

In particular,  $\deg_D R = 1$ ,  $\deg_D x = 2$ ,  $\deg_D S = 5$ ,  $\deg_D y = 6$  and  $\deg_D z = 10$ .

8.2.2. *The Module  $\mathcal{F}_{10}$ .* Define the submodule  $N_0$  of  $\mathcal{F}_{10}$  by:

$$N_0 = \mathcal{G}_{10}(R) = A + AR + AR^2 + AR^3 + AR^4 + AR^5 + AR^6 + AR^7 + AR^8 + AR^9 + AR^{10}$$

Since  $R^2 = xG - F^3$ , we see that:

$$N_0 \subset N_1 := A + AR + Ax + AxR + Ax^2 + Ax^2R + Ax^3 + Ax^3R + Ax^4 + Ax^4R + Ax^5 \subset \mathcal{F}_{10}$$

Since  $x^3 = R - yF$  we see that:

$$N_1 \subset N_2 := A + AR + Ax + AxR + Ax^2 + Ax^2R + Ay + Ax^3R + Axy + Ax^4R + Ax^2y \subset \mathcal{F}_{10}$$

Since  $x^2R = G - FS$  we see that:

$$N_2 \subset N_3 := A + AR + Ax + AxR + Ax^2 + AS + Ay + AxS + Axy + Ax^2S + Ax^2y \subset \mathcal{F}_{10}$$

Since  $x^2y = S - zF$  we see that:

$$N_3 \subset M := A + AR + Ax + AxR + Ax^2 + AS + Ay + AxS + Axy + Ax^2S + Az \subset \mathcal{F}_{10}$$

Note that  $F^2M \subset N_1 \subset M$ .

**Lemma 8.1.**  $FB \cap M = FM$

*Proof.* Let  $\pi_F : B \rightarrow B/FB$  be the canonical surjection. Let  $\bar{b} = \pi_F(b)$  for  $b \in B$ ,  $\bar{A} = \pi_F(A)$ ,  $\bar{M} = \pi_F(M)$  and  $\bar{B} = \pi_F(B)$ . Since  $F = xz - y^2$ , we see that:

$$(5) \quad k[\bar{x}, \bar{z}] = k^{[2]} \quad \text{and} \quad \bar{B} = k[\bar{x}, \bar{z}] \oplus k[\bar{x}, \bar{z}]\bar{y}$$

We have:

$$\bar{G} = \bar{x}^5, \quad \bar{R} = \bar{x}^3, \quad \bar{S} = \bar{x}^2\bar{y}, \quad \bar{A} = k[\bar{x}^5]$$

Define:

$$(6) \quad \mathcal{O} = k[\bar{x}] = \bar{A} \oplus \bar{A}\bar{x} \oplus \bar{A}\bar{x}^2 \oplus \bar{A}\bar{x}^3 \oplus \bar{A}\bar{x}^4$$

Then:

$$\begin{aligned} \bar{M} &= \bar{A} + \bar{A}\bar{x}^3 + \bar{A}\bar{x} + \bar{A}\bar{x}^4 + \bar{A}\bar{x}^2 + \bar{A}\bar{x}^2\bar{y} + \bar{A}\bar{y} + \bar{A}\bar{x}^3\bar{y} + \bar{A}\bar{x}\bar{y} + \bar{A}\bar{x}^4\bar{y} + \bar{A}\bar{z} \\ &= \mathcal{O} + \mathcal{O}\bar{y} + \bar{A}\bar{z} \end{aligned}$$

From (5) and (6) it follows that  $\bar{M}$  is a free  $\bar{A}$ -module of rank 11. Since  $FB \cap A = FA$  by (1), we conclude that  $FB \cap M = FM$ .  $\square$

**Lemma 8.2.**  $GB \cap M = GM$

*Proof.* Let  $\pi_G : B \rightarrow B/GB$  be the canonical surjection and let  $\bar{D}$  be the locally nilpotent derivation of  $B/GB$  induced by  $D$ . Let  $\bar{b} = \pi_G(b)$  for  $b \in B$ ,  $\bar{A} = \pi_G(A)$  and  $\bar{N}_1 = \pi_G(N_1)$ . Then  $\bar{A} = k[\bar{F}]$ . Define  $\mathcal{R} = k[\bar{F}, \bar{R}]$ . Since  $xG = F^3 + R^2$ , we have  $\bar{F}^3 + \bar{R}^2 = 0$ . In addition, note that  $\mathcal{R} \subset \ker \bar{D}$  and  $\bar{D}\bar{x} \neq 0$ . We thus have:

$$\mathcal{R} = \bar{A} \oplus \bar{A}\bar{R} \quad \text{and} \quad \mathcal{R}[\bar{x}] = \mathcal{R}^{[1]}$$

Therefore,

$$\begin{aligned} \bar{N}_1 &= \bar{A} + \bar{A}\bar{R} + \bar{A}\bar{x} + \bar{A}\bar{x}\bar{R} + \bar{A}\bar{x}^2 + \bar{A}\bar{x}^2\bar{R} + \bar{A}\bar{x}^3 + \bar{A}\bar{x}^3\bar{R} + \bar{A}\bar{x}^4 + \bar{A}\bar{x}^4\bar{R} + \bar{A}\bar{x}^5 \\ &= \mathcal{R} \oplus \mathcal{R}\bar{x} \oplus \mathcal{R}\bar{x}^2 \oplus \mathcal{R}\bar{x}^3 \oplus \mathcal{R}\bar{x}^4 \oplus \bar{A}\bar{x}^5 \end{aligned}$$

is a free  $\bar{A}$ -module of rank 11. Since  $GB \cap A = GA$  by (1), we conclude that:

$$(7) \quad GB \cap N_1 = GN_1$$

Suppose that  $Gw \in M$  for some  $w \in B$ . Then  $F^2Gw \in F^2M \subset N_1$ , so  $F^2Gw \in GB \cap N_1 = GN_1$  by (7). Therefore,  $F^2w \in N_1 \subset M$ , so  $F^2w \in F^2B \cap M = F^2M$  by Lemma 8.1, and  $w \in M$ .  $\square$

**Theorem 8.3.**  $M = \mathcal{F}_{10}$ .

*Proof.* By Thm. 4.4 (c), it will suffice to show  $FG \cdot B \cap M = FG \cdot M$ . This follows immediately from Lemma 8.1 and Lemma 8.2.  $\square$

Note that, by degree considerations,  $\mathcal{F}_{10}$  is a free  $A$ -module of rank 11. Therefore:

$$\begin{aligned}
\mathcal{F}_0 &= A \\
\mathcal{F}_1 &= A + AR \\
\mathcal{F}_2 &= A + AR + Ax \\
\mathcal{F}_3 &= A + AR + Ax + AxR \\
\mathcal{F}_4 &= A + AR + Ax + AxR + Ax^2 \\
\mathcal{F}_5 &= A + AR + Ax + AxR + Ax^2 + AS \\
\mathcal{F}_6 &= A + AR + Ax + AxR + Ax^2 + AS + Ay \\
\mathcal{F}_7 &= A + AR + Ax + AxR + Ax^2 + AS + Ay + AxS \\
\mathcal{F}_8 &= A + AR + Ax + AxR + Ax^2 + AS + Ay + AxS + Axy \\
\mathcal{F}_9 &= A + AR + Ax + AxR + Ax^2 + AS + Ay + AxS + Axy + Ax^2S \\
\mathcal{F}_{10} &= A + AR + Ax + AxR + Ax^2 + AS + Ay + AxS + Axy + Ax^2S + Az
\end{aligned}$$

8.2.3. *Degree Resolution.* Results above give the degree resolution of  $B$  induced by  $D$ .

- (1)  $B_0 = \mathcal{F}_0 = A = k[F, G] = k^{[2]}$
- (2)  $B_1 = k[\mathcal{F}_1] = A[R] = k[F, G, R] = k^{[3]}$
- (3)  $B_2 = k[\mathcal{F}_2] = B_1[x] = k[F, G, R, x]$  where  $xG = F^3 + R^2$
- (4)  $B_4 = B_3 = B_2$
- (5)  $B_5 = k[\mathcal{F}_5] = B_2[S] = k[F, R, x, S]$  where  $F(xS - F^2) = R(R - x^3)$
- (6)  $B_6 = k[\mathcal{F}_6] = B_5[y] = k[F, x, S, y]$  where  $x(S - x^2y) = F(F + y^2)$
- (7)  $B_9 = B_8 = B_7 = B_6$
- (8)  $B_{10} = k[\mathcal{F}_{10}] = B_6[z] = B$

It follows that  $\mathcal{N}_A(B) = \{n_0, \dots, n_5\} = \{0, 1, 2, 5, 6, 10\}$  and  $\text{index}_B(A) = 5$ . Observe that  $B_0, B_1, B_2, B_{10}$  are UFDs, whereas neither  $B_5$  nor  $B_6$  is a UFD.

8.2.4. *Fixed Points.* Let  $X_i = \text{Spec}(B_{n_i})$  for  $i = 0, \dots, 5$ .

- (1)  $X_1^{\mathbb{G}_a} = \mathcal{V}(FG) \subset X_1$ , which defines two planes in  $\mathbb{A}^3$ .
- (2)  $X_2^{\mathbb{G}_a} = \mathcal{V}(F) \subset X_2$ , which defines a cone.
- (3)  $X_3^{\mathbb{G}_a} = \mathcal{V}(F, R) \subset X_3$ , which defines a plane.
- (4)  $X_4^{\mathbb{G}_a} = \mathcal{V}(F, x) \subset X_4$ , which defines a plane.
- (5)  $X_5^{\mathbb{G}_a} = \mathcal{V}(x, y) \subset X_5$ , which defines a line in  $\mathbb{A}^3$ .

8.2.5. *Affine Modifications.* We describe each ring  $B_{n_{i+1}}$  as a  $\mathbb{G}_a$ -equivariant affine modification of  $B_{n_i}$ ,  $1 \leq i \leq 4$ .

- (1)  $B_2 = B_1[G^{-1}J_1]$  for  $J_1 = G \cdot B_1 + (F^3 + R^2) \cdot B_1$
- (2)  $B_5 = B_2[F^{-1}J_2]$  for  $J_2 = F \cdot B_2 + (G - x^2R) \cdot B_2$
- (3)  $B_6 = B_5[F^{-1}J_5]$  for  $J_5 = F \cdot B_5 + (R - x^3) \cdot B_5$
- (4)  $B_{10} = B_6[F^{-1}J_6]$  for  $J_6 = F \cdot B_6 + (S - x^2y) \cdot B_6$

8.2.6. *Canonical Factorization.* Let  $X = \operatorname{Spec}(B) = \mathbb{A}^3$  and  $Y = \operatorname{Spec}(A) = \mathbb{A}^2$ , and let  $\pi : X \rightarrow Y$  be the quotient morphism. Over points  $p \in Y$  defined by  $F = \alpha$ ,  $G = \beta$ , the fiber  $\pi^{-1}(p)$  is a line which is a single orbit if  $\alpha, \beta \neq 0$ ; a union of five lines which are orbits if  $\alpha = 0$ ,  $\beta \neq 0$ ; a union of two lines which are orbits if  $\alpha \neq 0$ ,  $\beta = 0$ ; and a line of fixed points if  $\alpha = \beta = 0$ .

Let  $\pi_i : X_{i+1} \rightarrow X_i$  be the morphism induced by the inclusion  $B_{n_i} \rightarrow B_{n_{i+1}}$ ,  $0 \leq i \leq 4$ . The canonical factorization of  $\pi$  is given by:

$$X = X_5 \xrightarrow{\pi_4} X_4 \xrightarrow{\pi_3} X_3 \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = Y$$

We consider each mapping  $\pi_i$  individually.

(1) For  $\pi_0$ , we have  $X_0 = \mathbb{A}^2$ ,  $X_1 = X_0 \times \mathbb{A}^1$  and  $\pi_0$  is projection on the first factor.

(2) For  $\pi_1$ , let  $W_1 = \mathcal{V}(G) \subset X_1$  and  $W_2 = \mathcal{V}(G) \subset X_2$ . Since  $(B_1)_G = (B_2)_G$ , the mapping

$$\pi_1 : X_2 \setminus W_2 \rightarrow X_1 \setminus W_1$$

is an isomorphism. We find that  $W_1 = \mathbb{A}^2$  and  $\pi_1(W_2) = C$ , where  $C$  is the cuspidal cubic curve  $\mathcal{V}(G, F^3 + R^2)$  in  $W_1$ , and that  $W_2 = C \times \mathbb{A}^1$ , where the restriction of  $\pi_1$  to  $W_2$  is projection on the first factor. The image of  $\pi_1$  excludes  $W_1 \setminus C$ .

(3) For  $\pi_2$ , let  $V_2 = X_2^{\mathbb{G}_a} = \mathcal{V}(F) \subset X_2$  and  $W_3 = \mathcal{V}(F) \subset X_3$ . Since  $(B_2)_F = (B_5)_F$ , the mapping

$$\pi_2 : X_3 \setminus W_3 \rightarrow X_2 \setminus V_2$$

is an isomorphism. We find that  $\pi_2(W_3) = Z$ , where  $Z$  is the union of two lines  $\mathcal{V}(F, G, R)$  and  $\mathcal{V}(F, G - x^5, R - x^3)$  in  $V_2$ , and that  $W_3 = Z \times \mathbb{A}^1$ , where the restriction of  $\pi_2$  to  $W_3$  is projection on the first factor. The image of  $\pi_2$  excludes  $W_2 \setminus Z$ .

(4) For  $\pi_3$ , let  $V_3 = X_3^{\mathbb{G}_a} = \mathcal{V}(F, R) \subset X_3$  and  $V_4 = X_4^{\mathbb{G}_a} = \mathcal{V}(F, x) = \mathcal{V}(F, R) \subset X_4$ . Since  $(B_5)_F = (B_6)_F$  and  $(B_5)_R = (B_6)_R$ , the mapping

$$\pi_3 : X_4 \setminus V_4 \rightarrow X_3 \setminus V_3$$

is an isomorphism. We find that  $\pi_3(V_4) = L$ , where  $L$  is the line  $\mathcal{V}(F, R, x) \subset X_3$ , and that  $V_4 = L \times \mathbb{A}^1$  (a plane), where the restriction of  $\pi_3$  to  $V_4$  is projection on the first factor. The image of  $\pi_3$  excludes  $V_3 \setminus L$ .

(5) For  $\pi_4$ , let  $X_5^{\mathbb{G}_a} = \mathcal{V}(F, x) = \mathcal{V}(x, y) \subset X_5$ . Since  $(B_6)_F = B_F$  and  $(B_6)_x = B_x$ , the mapping

$$\pi_4 : X_5 \setminus V_5 \rightarrow X_4 \setminus V_4$$

is an isomorphism. We find that  $\pi_4(V_5) = P$ , where  $P$  is the point  $\mathcal{V}(F, x, S, y)$ , and that  $V_5 = P \times \mathbb{A}^1$  (a line), where the restriction of  $\pi_4$  to  $V_5$  is projection on the first factor. The image of  $\pi_4$  excludes  $V_4 \setminus P$ .

The affine modification  $\pi_4 : X_5 \rightarrow X_4$  differs from the first three in that its exceptional locus is one-dimensional. One way to understand this situation is to view  $X_4$  as a subvariety of  $\mathbb{A}^4$  given by  $xT = F(F + y^2)$  in coordinates  $x, F, y, T$  (where  $T = S - x^2y$ ). According to *Prop. 2.1* of [11], we can view  $\pi_4$  as the restriction of the affine modification of  $\mathbb{A}^4$  along the divisor  $\{F = 0\}$  with center  $\{F = T = 0\}$ . If  $\beta : \mathcal{X} \rightarrow \mathbb{A}^4$  is the associated morphism, then  $\mathcal{X} \cong \mathbb{A}^4$  with coordinates  $x, F, y, \frac{T}{F}$ ; and  $X_5 \subset \mathcal{X}$  is the hyperplane defined by  $x\frac{T}{F} = F + y^2$ .

8.2.7. *A D-Basis for B.* Let  $p : B \rightarrow B/zB = k[x, y] \cong k^{[2]}$  be the standard surjection, and set  $\bar{A} = p(A)$  and  $\bar{\mathcal{F}}_n = p(\mathcal{F}_n)$ .

**Proposition 8.4.**  $\bar{\mathcal{F}}_9 = k[x, y]$  and  $\bar{\mathcal{F}}_9$  is a free  $\bar{A}$ -module of rank 10.

*Proof.* We have  $\bar{A} = k[y^2, x^5 + 2x^2y^3]$ . We see that  $\bar{A} \subset \bar{A}[y]$  is an integral extension of degree 2, and that  $\bar{A}[y] \subset k[x, y]$  is an integral extension of degree 5. Therefore,  $k[x, y]$  is a free  $\bar{A}$ -module of rank 10 with basis:

$$\mathcal{B} = \{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y\}$$

Observe that  $\mathcal{F}_9$  contains the set

$$\mathcal{C} = \{1, x, x^2, R - yF, x(R - yF), y, xy, S, xS, x^2S\}$$

and that  $p(\mathcal{C}) = \mathcal{B}$ . Therefore,  $\bar{\mathcal{F}}_9 = k[x, y]$ .  $\square$

**Corollary 8.5.**  $B = \bigoplus_{i \geq 0} \mathcal{F}_9 \cdot z^i$

*Proof.* Set  $N = \sum_{i \geq 0} \mathcal{F}_9 \cdot z^i$ . By Lemma 8.2, we see that  $N = \bigoplus_{i \geq 0} \mathcal{F}_9 \cdot z^i$ .

Consider the descending chain of submodules:

$$B \supset N + zB \supset N + z^2B \supset \dots$$

By the proposition, we see that  $B = N + zB$ . Since  $N + zN = N$ , it follows that  $N + z^nB = B$  for every  $n \geq 0$ . Given nonzero  $f \in B$ , choose an integer  $n > \deg_D f$  and write  $f = \sum_{0 \leq i \leq n-1} a_i z^i + z^n b$ ,  $a_i \in \mathcal{F}_9$  and  $b \in B$ . If  $b \neq 0$ , then  $\deg_D(a_i z^i) < \deg_D(z^n b)$  for  $0 \leq i \leq n-1$ , which implies

$$\deg_D f = \deg_D(z^n b) = 10n + \deg b$$

a contradiction. Therefore,  $b = 0$  and  $f \in N$ .  $\square$

It is shown above that  $\mathcal{F}_9$  is a free  $A$ -module which admits a  $D$ -basis. Thus, the equality in Cor. 8.5, together with Lemma 8.2(b) and the  $D$ -basis of  $\mathcal{F}_9$ , give a  $D$ -basis for each  $\mathcal{F}_n$ ,  $n \geq 0$ . It follows that  $B$  is a free  $A$ -module which admits a  $D$ -basis.

8.2.8. *Associated Graded Ring.* The foregoing calculations show the following.

$$\text{Gr}_D(B) = A[FGt, F^2Gt^2, F^4G^3t^5, F^5G^3t^6, F^8G^5t^{10}] \subset A[t] = A^{[1]}$$

In particular,  $\text{Gr}_D(B)$  is finitely generated as a  $k$ -algebra.

8.3. **Russell Cubic Threefold.** (See also [11], Examples 1.5, 3.2.) Let  $\mathbb{C}[x, y, z, t] = \mathbb{C}^{[4]}$ . The Russell cubic threefold  $X \subset \mathbb{C}^4$  is defined by the zero set of the polynomial  $x + x^2y + z^2 + t^3$ .  $X$  is smooth, contractible and factorial, and  $X$  is diffeomorphic to  $\mathbb{R}^8$ . On the other hand, the function  $x$  restricted to  $X$  is an invariant of every  $\mathbb{G}_a$ -action on  $X$ , which implies that  $X$  is not isomorphic to  $\mathbb{C}^3$  as a complex algebraic variety; see, for example, [10].  $X$  is an example of an exotic affine space.

Let the coordinate ring  $B = \mathbb{C}[X]$  be given by  $B = \mathbb{C}[x, y, z, t]$ , where  $x + x^2y + z^2 + t^3 = 0$ . The derivation  $D = x^2 \frac{\partial}{\partial z} - 2z \frac{\partial}{\partial y}$  of  $B$  is locally nilpotent, and if  $A = \ker D$ , then  $A = \mathbb{C}[x, t]$ .

**Lemma 8.6.**  $\text{pl}(D) = x^2A$

*Proof.* Let  $\mathcal{F}_1$  be the first degree module for  $D$ , noting that  $A + Az \subset \mathcal{F}_1$ .

Let  $\bar{B} = B/xB$  and let  $\bar{D}$  be the locally nilpotent derivation on  $\bar{B}$  induced by  $D$ . Then:

$$\bar{B} = \frac{\mathbb{C}[z, t]}{(z^2 + t^3)}[y] \quad \text{and} \quad \ker \bar{D} = \frac{\mathbb{C}[z, t]}{(z^2 + t^3)} = \mathbb{C}[t] \oplus \mathbb{C}[t] \cdot z$$

If  $g \in \mathcal{F}_1$  and  $xg \in A + Az$ , then  $Dg \in xA$ . Write  $Dg = xP(t) + x^2h$  for some  $P \in \mathbb{C}[t]$  and  $h \in A$ .<sup>1</sup> Then  $g - zh \in A + Az$  and  $D(g - zh) = xP(t)$ . Since  $g - zh, z \in \mathcal{F}_1$ , we have:

$$(g - zh)x^2 - xP(t)z \in A \quad \Rightarrow \quad (g - zh)x - P(t)z \in A$$

Modulo  $xB$ , this implies:

$$-P(t)z \in \bar{A} = \mathbb{C}[t] \subset \ker \bar{D} = \mathbb{C}[t] \oplus \mathbb{C}[t] \cdot z \quad \Rightarrow \quad P(t) \equiv 0 \quad \Rightarrow \quad P(t) \in xB$$

Since  $\mathbb{C}[x, t] \cong \mathbb{C}^{[2]}$ , it follows that  $P(t) = 0$  and  $D(g - zh) = 0$ . Therefore,  $g \in A + Az$ . By Thm. 4.4,  $\mathcal{F}_1 = A + Az$ , which implies  $\text{pl}(D) = D\mathcal{F}_1 = x^2A$ .  $\square$

<sup>1</sup> $Dg$  has no constant term, since 0 is a fixed point of the  $\mathbb{G}_a$ -action.

By this lemma,  $B_1 = A[z] = \mathbb{C}[x, z, t] \cong \mathbb{C}^{[3]}$ . In addition, since  $\deg_D(y) = 2$ ,  $B_2 = B_1[y] = B$ . We find that  $B_2 = B_1[x^{-2}I]$ , where  $I = x^2B_1 + (x + z^2 + t^3)B_1$ .

Let  $X_1 = \text{Spec}(B_1) \cong \mathbb{C}^3$ , and let  $Y = \text{Spec}(A)$ . The canonical resolution of the quotient morphism  $\pi : X \rightarrow Y$  is given by:

$$X = X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = Y$$

We have that  $X_1 = Y \times \mathbb{C}^1$ , and  $\pi_0$  is projection on the first factor.

Let  $W_1 = \mathcal{V}(x) \subset X_1$  and  $W = \mathcal{V}(x) \subset X$ . Then  $W_1 \cong \mathbb{C}^2$  and  $W \cong C \times \mathbb{C}^1$ , where  $C$  is the cuspidal cubic curve  $\mathcal{V}(x, z^2 + t^3)$ . Since  $(B_1)_x = B_x$ ,  $\pi_1 : X \setminus W \rightarrow X_1 \setminus W_1$  is an isomorphism. Otherwise,  $\pi_1(W) = C$ .

**8.4. Winkelmann's Example.** In [14], Winkelmann gave the first examples of free  $\mathbb{G}_a$ -actions on affine space which are not translations. We analyze the smallest of these examples, which is in dimension four.

Let  $B = k[x, y, z, u] = k^{[4]}$ , and define  $F \in B$  by  $F = 2xz - y^2$ . Define  $D \in \text{LND}(B)$  by:

$$Dx = 0, \quad Dy = x, \quad Dz = y, \quad Du = F + 1$$

Let  $A = \ker D$ , and let  $\pi : X \rightarrow Y$  be the induced quotient morphism, where  $X = \text{Spec}(B)$  and  $Y = \text{Spec}(A)$ . Since  $xB + yB + (F + 1)B = B$ , the induced  $\mathbb{G}_a$ -action on  $\mathbb{A}^4$  is fixed-point free.

Let  $\mathcal{F}_n$  be the  $A$ -module  $\mathcal{F}_n = \ker D^{n+1}$  and let  $B_n = k[\mathcal{F}_n]$ ,  $n \geq 0$ . Since  $x, y, z, u \in \mathcal{F}_2$ , we see that  $B = B_2$ , so the degree resolution of  $A$  is given by:

$$A = B_0 \subset B_1 \subset B_2 = B$$

Hence, if  $X_i = \text{Spec}(B_i)$ , then the canonical factorization of  $\pi$  is given by:

$$X = \mathbb{A}^4 = X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = Y$$

**8.4.1. The Plinth Ideal.** Observe the following.

- (1) Since  $y$  and  $u$  are local slices,  $G := uDy - yDu \in A$ .
- (2) If  $T = yu - 2z(F + 1)$ , then  $DT = G$ .
- (3) Since  $u$  and  $T$  are local slices,  $H := uDT - TDu \in A$ .
- (4) It is easy to show that  $A = k[x, F, G, H]$ . The prime relation for this ring is:

$$xH = G^2 + F(F + 1)$$

**Lemma 8.7.**  $\text{pl}(D) = (x, F + 1, G)$ .

*Proof.* Let  $\sigma : B \rightarrow B/xB$  be the standard surjection, let  $\bar{A} = \sigma(A)$  and let  $\bar{H} = \sigma(H)$ . Then:

$$\bar{A} = k[y^2, y(y^2 - 1), \bar{H}] = k[y^2, y(y^2 - 1)]^{[1]}$$

Let  $J \subset A$  be the ideal  $J = (x, F + 1, G)$ , and let  $f \in I_1$  be given. Since  $J \subset I_1$  and  $A/J = k[H]$ , it suffices to assume  $f \in k[H]$ .

Write  $f = P(H)$  for  $P \in k^{[1]}$  and let  $L \in B$  be such that  $DL = f$ . Then:

$$yDL - LDy = yP(H) - xL \in A \quad \Rightarrow \quad yP(\bar{H}) \in \bar{A}$$

If  $P(\bar{H}) \neq 0$ , choose  $\lambda \in k$  so that  $P(\lambda) \neq 0$ . Then:

$$yP(\lambda) \in k[y^2, y(y^2 - 1)] \quad \Rightarrow \quad k[y^2, y(y^2 - 1)] = k[y]$$

But this is clearly a contradiction. Therefore,  $P(\bar{H}) = 0$ , which implies  $f = P(H) = 0$ . Therefore,  $J = I_1$ .  $\square$



8.4.2. *The mapping  $\pi_0$ .* Let  $W_0 \subset Y$  be defined by  $W_0 = \text{Spec}(A/I_1)$ . Lemma 8.7 implies that  $A/I_1 = k[H] \cong k^{[1]}$ , so  $W_0 \cong \mathbb{A}^1$ . Let  $V = Y \setminus W_0$ . Then there is an open set  $U \subset X_1$  such that  $U = V \times \mathbb{A}^1$  and  $\pi_0$  is projection on the first factor. In particular, let  $D_1$  be the restriction of  $D$  to  $B_1$ . Since  $y, u, T$  are local slices, we have:

$$(B_1)_x = A_x[y], \quad (B_1)_{F+1} = A_{F+1}[u], \quad (B_1)_G = A_G[T]$$

Therefore,  $U = U_x \cup U_{F+1} \cup U_T$ , where  $U_x = \{x \neq 0\}$ ,  $U_{F+1} = \{F+1 \neq 0\}$  and  $U_T = \{T \neq 0\}$ .

Lemma 8.7 further implies that  $\mathcal{F}_1 = A + Ay + Au + AT$ . Relations in this module are given by:

$$yG - xT = F(F+1), \quad xu - y(F+1) = G, \quad uG - (F+1)T = H$$

The ring  $B_1$  is given by  $B_1 = k[x, y, u, F, N]$ , where  $N = z(F+1)$ . The prime relation in this ring is  $2xN = (F+y^2)(F+1)$ .

The closed set  $W_1 = X_1 \setminus U$  is the set of fixed points  $X_1^{\mathbb{G}_a}$ , defined by the ideal:

$$I_1 B_1 = xB_1 + (F+1)B_1$$

We find that  $W_1 \cong \mathbb{A}^3$ . Therefore,  $\pi_0$  restricts to  $\pi_0 : W_1 \cong \mathbb{A}^3 \rightarrow W_0 \cong \mathbb{A}^1$  and we find that  $\pi_0(W_1) = p$ , where  $p \in Y$  is the point defined by the ideal  $(x, F+1, G, H) \subset A$ . Therefore,  $\pi_0^{-1}(W_0 \setminus \{p\}) = \emptyset$ .

8.4.3. *The mapping  $\pi_1$ .* Let  $W \subset X = \mathbb{A}^4$  be defined by the ideal  $xB + (F+1)B$ . Then  $W = W_+ \cup W_-$ , where  $W_+ = \{x=0, y=1\} \cong \mathbb{A}^2$  and  $W_- = \{x=0, y=-1\} \cong \mathbb{A}^2$ . Since  $(B_1)_x = B_x$  and  $(B_1)_{F+1} = B_{F+1}$ , we see that  $\pi_1$  maps  $X \setminus W$  isomorphically to  $X_1 \setminus W_1$ .

Using coordinate functions  $(y, u, N)$  on  $W_1 = \mathbb{A}^3$ , we find that  $\pi_1(W_+) = L_+ \cong \mathbb{A}^1$  and  $\pi_1(W_-) = L_- \cong \mathbb{A}^1$ , where  $L_+ = \{y=1, N=0\}$  and  $L_- = \{y=-1, N=0\}$ .

8.4.4. *Summary.* The canonical factorization of  $\pi$  splits as follows:

$$X \setminus W \xrightarrow[\sim]{\pi_1} X_1 \setminus X_1^{\mathbb{G}_a} \cong V \times \mathbb{A}^1 \xrightarrow{\pi_0} V$$

and:

$$W \cong \mathbb{A}^2 \cup \mathbb{A}^2 \xrightarrow{\pi_1} X_1^{\mathbb{G}_a} \cong \mathbb{A}^3 \xrightarrow{\pi_0} W_0 \cong \mathbb{A}^1, \quad \pi_1(W) = \mathbb{A}^1 \cup \mathbb{A}^1 \text{ and } \pi_0(X_1^{\mathbb{G}_a}) = p$$

As an affine modification, we find that  $B = B_1[(F+1)^{-1}J]$ , where  $J = (F+1)B_1 + NB_1$ .

## 9. A TRIANGULAR $R$ -DERIVATION OF $R^{[3]}$ WITH A SLICE

9.1. **The Derivation  $\delta$  of  $\mathbb{C}^{[4]}$ .** Let  $R = \mathbb{C}[x, y] = \mathbb{C}^{[2]}$  and  $B = R[z, u] = R^{[2]}$ , and define  $p, v \in B$  by  $p = yu + z^2$  and  $v = xz + yp$ . Define the triangular  $R$ -derivation of  $B$  by:

$$\delta = v_u \partial_z - v_z \partial_u = y^2 \partial_z - (x + 2yz) \partial_u$$

If  $A = \ker \delta$ , then  $A = R[v]$ . Let  $\{\mathcal{F}_n\}_{n \geq 0}$  be the degree modules for  $\delta$ . Since  $v, z \in \mathcal{F}_1$ , we see that  $p \in \mathcal{F}_1$ . Modulo  $y$ , we have  $p \equiv z^2$  and  $v \equiv xz$ , meaning that  $xp - vz = yq$  for some  $q \in \mathcal{F}_1$ . We find that  $q = xu - zp$ .

**Proposition 9.1.** *For the derivation  $\delta$ :*

- (a)  $\mathcal{F}_1 = A + Az + Ap + Aq$
- (b) *A complete set of  $A$ -relations for  $\mathcal{F}_1$  is given by:*

$$xz + yp - v = 0 \quad \text{and} \quad vz - xp + yq = 0$$

- (c)  $\text{pl}(\delta) = y^2 A + xyA + (x^2 + yv)A$

*Proof.* Define  $M \subset \mathcal{F}_1$  by  $M = A + Az + Ap + Aq$ . Since  $yp = v - xz$  and  $yq = xp - vz$ , we have:

$$M = A + \mathbb{C}[x, v]p + Az + \mathbb{C}[x, v]q$$

Suppose that  $F \in \mathcal{F}_1$  and  $xF \in M$ . Write:

$$xF = a_0 + a_1p + a_2z + a_3q \quad \text{where} \quad a_0, a_2 \in A, \quad a_1, a_3 \in \mathbb{C}[x, v]$$

Modulo  $x$ , we have  $0 = \bar{a}_0 + \bar{a}_1p + \bar{a}_2z - \bar{a}_3zp$  where  $\bar{a}_0, \bar{a}_2 \in \mathbb{C}[y, yp]$  and  $\bar{a}_1, \bar{a}_3 \in \mathbb{C}[yp]$ . Since  $\mathbb{C}[y, yp, z] \cong \mathbb{C}^{[3]}$ , it follows that  $\bar{a}_0 + \bar{a}_1p = \bar{a}_2 - \bar{a}_3p = 0$ . Since

$$\mathbb{C}[y, yp] + \mathbb{C}[yp]p = \mathbb{C}[y, yp] \oplus \mathbb{C}[yp]p$$

we conclude that  $\bar{a}_i = 0$  for each  $i$ . If  $a_i = xb_i$  for  $b_i \in B$ , then  $b_i \in A$ , since  $A$  is factorially closed. Therefore,  $F \in M$ . By *Thm. 4.4*, it follows that  $M = \mathcal{F}_1$ . This proves part (a), and the same argument shows part (b).

Since  $\text{pl}(\delta) = \delta\mathcal{F}_1$ , part (c) follows by the observing that:

$$\delta z = y^2, \quad \delta p = -xy, \quad \delta q = -(x^2 + yv)$$

□

The **Vénèreau polynomial**  $f \in A$  is defined by  $f = y + xv$ . It is well known that  $\mathcal{V}(f) \cong \mathbb{C}^3$ , but it is not known whether  $f$  is a variable of  $B$ . See [6].

Observe that, if  $r \in \mathcal{F}_1$  is defined by  $r = yz + (v^4 - 3fv)p - xv^3q$ , then  $\delta r = f^3$ .

**Corollary 9.2.**  $f^2 \notin \text{pl}(\delta)$

*Proof.* Let  $\mathbb{C}[X, Y, Z] = \mathbb{C}^{[3]}$  and  $I = (Y^2, XY, X^2 + YZ)$ . It must be shown that  $(Y + XZ)^2 \notin I$ .

Let  $\mathbb{C}[X, Y, Z] = \bigoplus_{i \geq 0} V_i$  be the standard  $\mathbb{Z}$ -grading of  $\mathbb{C}[X, Y, Z]$ , where  $V_i$  is the vector space of homogeneous forms of degree  $i$ . Then  $I = \bigoplus_{i \geq 0} I_i$  is a graded ideal, where  $I_i = I \cap V_i$ .

Let  $\mathcal{W} = Y^2 \cdot V_2 + XY \cdot V_2$ . Then:

$$I_4 = \mathcal{W} + (X^2 + YZ) \cdot V_2 = \mathcal{W} + \mathbb{C} \cdot (X^2 + YZ)Z^2$$

Therefore,  $\dim_{\mathbb{C}} I_4/\mathcal{W} = 1$ . Note that the elements  $X^2Z^2$  and  $YZ^3$  are linearly independent modulo  $\mathcal{W}$ , since no element of  $\mathcal{W}$  has a term supporting  $X^2Z^2$  or  $YZ^3$ . Since  $(X^2 + YZ)Z^2 \in I_4$ , it follows that  $X^2Z^2 \notin I_4$ .

If  $(Y + XZ)^2 \in I$ , then  $X^2Z^2 \in I_4$ , a contradiction. Therefore,  $(Y + XZ)^2 \notin I$ . □

Extend  $\delta$  to the triangular derivation  $\Delta$  on  $B[t] = B^{[1]}$  by  $\Delta t = 1 + f + f^2$ . If  $s = (1 - f)t + r$ , then  $\Delta s = 1$ . The kernel of  $\Delta$  is the image of the Dixmier map induced by  $s$ , namely:

$$\ker \Delta = R[z - y^2s, u + (x + 2yz)s - y^3s^2, f^3t - (1 + f + f^2)r]$$

According to [7],  $\ker \Delta$  is a  $\mathbb{C}^2$ -fibration over  $R$ . The question is whether it is a trivial fibration.

**Question 9.3.** *Do there exist  $P, Q \in B[t]$  with  $\ker \Delta = R[P, Q]$ ?*

## 10. CONCLUSION

**10.1. Module Generators.** In *Thm. 4.4*, for the initial module  $M_0 \subset \mathcal{F}_n$ , we can always take  $M_0 = \mathcal{G}_n(r)$  for a local slice  $r$ . If  $n \geq 2$  and  $\mathcal{F}_1, \dots, \mathcal{F}_{n-1}$  are known, then a more efficient choice is:

$$M_0 = \sum_{1 \leq i \leq n-1} \mathcal{F}_i \mathcal{F}_{n-i}$$

We ask the following: If  $B = B_N = k[\mathcal{F}_N]$ , does it follow that, for each  $n \geq 0$ ,

$$\mathcal{F}_n = \sum_{e_1 + 2e_2 + \dots + Ne_N = n} \mathcal{F}_1^{e_1} \cdots \mathcal{F}_N^{e_N} \quad (\text{where } \mathcal{F}_i^0 = A) ?$$

More generally, is  $\text{Gr}_D(B)$  finitely generated as a  $k$ -algebra (given that  $B$  is affine)?

**10.2. Related Work.** In his thesis, Alhajjar also gives an algorithm for finding degree modules  $\mathcal{F}_n$  associated to a locally nilpotent derivation. His algorithm is very different than the one presented herein, employing what he terms a “twisted embedding technique”. The algorithm we give is modeled after the algorithm of van den Essen for finding generators of the kernel of a locally nilpotent derivation, i.e., the ring of invariants of the corresponding  $\mathbb{G}_a$ -action. Van den Essen’s algorithm, in turn, is a generalization of the technique used by Tan to find generators for the invariants of an irreducible representation of  $\mathbb{G}_a$ , a technique which was essentially already in use in the Nineteenth Century. See [1, 4, 13].

For fixed integer  $d \geq 0$ , Alhajjar defines the invariant subring  $AL_d(X) = \cap_D k[\mathcal{F}_d]$ , where  $D$  ranges over all locally nilpotent derivations of  $k[X]$ . These rings generalize the well-known Makar-Limanov invariant of  $k[X]$ . Alhajjar’s goal is to study affine  $k$ -varieties  $X$  which are semi-rigid, meaning that  $X$  admits an essentially unique non-trivial  $\mathbb{G}_a$ -action. In this case,  $AL_d(X) = k[\mathcal{F}_d]$  for the non-trivial action, and he studies the sequence of inclusions  $AL_d(X) \subset AL_{d+1}(X)$ .

**10.3. Remark.** If  $K$  is an algebraically closed field of positive characteristic, then any  $\mathbb{G}_a$ -action on  $\mathbb{A}_K^n$  induces a degree function, hence a filtration, on the coordinate ring  $R = K^{[n]}$ , where elements of the invariant ring  $R^{\mathbb{G}_a}$  have degree zero. See, for example, [2]. In this way, one obtains a canonical factorization of the quotient morphism for such an action, under the assumption that  $R^{\mathbb{G}_a}$  is affine over  $K$ .<sup>2</sup>

**10.4. The Freeness Conjecture.** Let  $B = k^{[3]}$ . Given nonzero  $D \in \text{LND}(B)$  with  $A = \ker D$ , Miyanishi’s Theorem asserts that  $A \cong k^{[2]}$ . We make the following.

**Conjecture.** *Let  $B = k^{[3]}$ . Given  $D \in \text{LND}(B)$ , if  $A = \ker D$ , then the following equivalent conditions hold.*

1.  *$B$  is a free  $A$ -module with basis  $\{Q_i\}_{i \geq 0}$  such that  $\deg_D Q_i = i$ .*
2. *Each degree module  $\mathcal{F}_n$  is a free  $A$ -module with basis  $\{Q_i\}_{0 \leq i \leq n}$  such that  $\deg_D Q_i = i$ .*
3. *Each image ideal  $I_n = D^n \mathcal{F}_n \subset A$  is principal.*

In addition to the examples above, evidence for this conjecture includes the following.

- (a)  $\mathcal{F}_1 = A \oplus Ar$  for a local slice  $r$ .
- (b) By combining Miyanishi’s Theorem with the Quillen-Suslin Theorem, Daigle has shown that  $\mathcal{F}_n$  is a free  $A$ -module of rank  $n + 1$  for each  $n \geq 0$  (unpublished).

What are the geometric implications of this conjecture? Given  $D \in \text{LND}(B)$ , let

$$A = B_0 \subset B_{n_1} \subset \cdots \subset B_{n_r} = B \quad \text{and} \quad X = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \rightarrow \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = Y$$

be the degree resolution of  $B$  and canonical factorization of  $\pi$ , respectively, induced by  $D$ . If the Freeness Conjecture holds for  $D$ , then  $B_{n_i} = B_{n_{i-1}}[Q_{n_i}]$  for each  $i$ ,  $2 \leq i \leq r$ , meaning that each mapping  $\pi_i : X_{i+1} \rightarrow X_i$  is a *principal* affine modification ( $1 \leq i \leq r - 1$ ). By *Lemma 3.5*, the defining relation for this extension is of the form  $f^m Q_{n_i} = g$  for some  $f \in A$ ,  $g \in B_{n_{i-1}}$  and  $m \geq 1$ .

Moreover, observe that, in the examples of *Sect. 8*, we saw that  $B_{n_{i-1}} = k[x_1, x_2, x_3, x_4]$  for some  $x_j \in B$ , where  $x_1 \in A$  and  $x_1 Q_{n_i} = x_4$ . Consequently  $B_{n_i} = [x_1, x_2, x_3, Q_{n_i}]$ , meaning that, for these examples, each intermediate threefold  $X_i$  is a hypersurface in  $\mathbb{A}^4$ .

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<sup>2</sup> It is unknown whether this assumption is necessary, i.e., it is an open question whether  $R^{\mathbb{G}_a}$  is always finitely generated when  $K$  is of positive characteristic, even in the linear case.

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